Real numbers

1) **Motivation.**
Rational numbers form a field. Problem: some very well defined numbers, such as $\sqrt{2}$ and π, are not rational. They can not be *exactly* represented as $^p/_q$, but surely can be *approximated* by rational numbers with *arbitrary* precision. We can write their decimal expansion with arbitrary precision. So, this will be a motivation for our definition.
Remark: we could have used expansion in any base, but we are all more familiar with decimals.
2) **Definition.** $\mathbb{R}_+$ is the set of all infinite sequences $x_0, x_1, \dots$ (i.e. maps $x: \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$), such that $x_n \in \{0, 1, \dots, 9\}$ when $n > 0$***, facthored*** by the following equivalence relation: the sequence $x_0, x_1, \dots, x_n, 0,0, \dots$ is equivalent to $x_0, x_1, \dots, x_n - 1,9,9, \dots$ if $x_n \neq 0$.
The first *n* digits of the decimal representation *of x* guarantee that
$x_0 + 10^{-1}\, x_1 + \ \dots 10^{-n} x_n \leq x \leq x_0 + 10^{-1}\, x_1 + \ \dots 10^{-n} x_n + 10^{-n}$
Definition.$\mathbb{R}$ is $\{+,-\} \times \mathbb{R}_+/\sim$, where the only two equivalent elements are $(+, 0.0000 \dots)$ and $(-, 0.0000 \dots)$.
Remark. This definition differs from the book!
Graphic representation on the line also helps.
Let us understand the equivalency.
$1 = 1{,}000 \dots = 0.999 \dots$
$1 = 0 + 9 \times 10^{-1} + \ \dots 9 \times 10^{-n} + \cdots$ by the familiar formula for te sum of infinite geometric series.
We are not using an undefined notion here, this is just an explanation.
Rational numbers do not necessarily have representation ending with zeroes.
Examples:
$^1/_3 = 0.3333 \dots$
$^1/_6 = 0.166666 \dots$
$^1/_5 = 0.20000 \dots = 0.19999 \dots$
$\sqrt{2}$ =1.4142135623730950488016887… -- no law whatsoever!
Observation: there is a natural lexicographic order on $\mathbb{R}$. Note that the order is reversed for the negative numbers!
Easy to identify integers in $\mathbb{R}$. Also easy to define multiplication by 10^k: it is just a shift by k digits to the right and using the decimal expansion of the integers.

Real numbers -- continued

Theorem.
$x \in \mathbb{Q}$ iff $\exists N, d : x_{n+d} = x_n$ for $n > N$- the decimal expansion *is eventually periodic.*
Proof.
First, assume *x* has eventually periodic expansion. Then
$a := 10^{N+d}x - 10^N x$ has all zero decimal digits after period, so it is an integer. Which means
$x = \frac{a}{10^{N+d} - 10^N}$ is a rational number.
On the other hand, let $x = {}^p/_q$. Two of the numbers $1, 10, 10^2, \ldots, 10^q$ are the same $mod\ q$, because $\mathbb{Z}_q$ contain only *q* elements. So, say, $10^{N+d} \equiv 10^N (mod\ q)$, or $\exists m \in \mathbb{N} : qm = 10^{N+d} - 10^N$.
This means that
$$x = {}^p/_q = {}^{pm}/_{qm} = {}^{pm}/_{10^{N+d} - 10^N} = 10^{-N}\left({}^{pm}/_{10^d - 1}\right).$$
Now divide pm by $10^d - 1$ to obtain that for some integers $a < 10^d - 1$ and b
$$x = 10^{-N}\left({}^{a}/_{10^d - 1} + b\right).$$
If we write the decimal expansion of $a = a_1 a_2 \ldots a_d$ with exactly d digits, possibly putting $0's$ in front, we obtain the desired decimal expansion, by the equation above.
Note: we did not use the formula for the sum of the geometric series!
□
Using the order relation, we can define addition, multiplication, division, and check, that $\mathbb{R}$ is a field.
Long and boring, so we will skip it.
More interesting:
Absolute value: $|x| = \max\{x, -x\} = \begin{cases} x, x \geq 0 \\ -x, x \leq 0 \end{cases}.$
It can be used to define *distance* between two numbers: $|x - y|$.
Archimedian property: let $x > 0, y > 0$, then $\exists n \in \mathbb{N} : nx > y$.
Proof.
x has a nonzero decimal digit, so $x > 10^{-N}$ for some $N \in \mathbb{N}$. So we can take
$n = 10^N(y_0 + 1)$.
□

Limit of a sequence.

1) **Definition**. Let (a_n) be a sequence of real numbers. We say that the sequence converges to $L \in \mathbb{R}$,
$\lim_{n\to\infty} a_n = L$
iff $\forall \varepsilon > 0\ \exists N = N(\varepsilon): n > N \Rightarrow |a_n - L| < \varepsilon$.
The same as all but finitely many fall into ε –neighborhood.
Yet another view: Let us consider the residual set $S_N = \{a_n, n > N\}$.
Then $\forall \varepsilon > 0\ \exists N = N(\varepsilon): S_N \subset [L-\varepsilon, L+\varepsilon]$.
Remark: Real sequences can *diverge to* $\pm\infty$. $a_n \to \pm\infty$ iff
$\forall \varepsilon > 0\ \exists N = N(\varepsilon): n > N \Rightarrow a_n > \frac{1}{\varepsilon}\ (a_n < -\frac{1}{\varepsilon})$.
2) **Examples.** How to prove the convergence from the definition? Play the "$\varepsilon - N$ game": given a number $\varepsilon > 0$, find N. No need to find the *best N:* anything would be sufficient.
   - $\lim_{n\to\infty} L = L$.
   - $\lim_{n\to\infty} {}^{1}/_{n} = 0$. Follows from the Archimedean property.
   - $\lim_{n\to\infty} \frac{\cos n}{n^2+n} = 0$.
   - $\lim_{n\to\infty} \cos(n\pi)$ does not exist.
3) **An important tool:**
Squeezed sequence Theorem.
Let $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, and for some N, $a_n \le b_n \le c_n$ whenever $n > N$.
Then $\lim_{n\to\infty} b_n = L$.
Proof.
Fix $\varepsilon > 0$. Find two numbers N_a and N_c, such that
$n > N_a \Rightarrow |a_n - L| < \varepsilon, n > N_c \Rightarrow |c_n - L| < \varepsilon$
These numbers exist by the definition of the limit.
Take $N_b = \max(N_a, N_c, N)$.
Then for $n > N_b$ we have $|b_n - L| \le \max(|a_n - L|, |c_n - L|) < \varepsilon$
□
4) **Example:** $\lim_{n\to\infty} \frac{n^2+1}{n^3-n^2-1} = 0$.

Basic operations with limits.

Definition.

- $S \subset \mathbb{R}$ is *bounded above* if $\exists M \in \mathbb{R}$: $\forall s \in S$: $s \le M$. Any such M is called an *upper bound* for S.
- $S \subset \mathbb{R}$ is *bounded below* if $\exists L \in \mathbb{R}$: $\forall s \in S$: $L \le s$. Any such L is called a *lower bound* for S.
- $S \subset \mathbb{R}$ is called *bounded* if it is bounded both above and below.

Easy property. Subset of a set bounded (above, below) is bounded (above, below). Union and intersection of two bounded(above, below) sets is bounded(above, below). Any finite set is bounded.

Lemma.

Any converging sequence is bounded.

Proof.

A converging sequence is a union of a subset of $(L-1, L+1)$ and a finite set of elements which does not belong there.

Monotonicity Lemma.

Let $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$. Assume that $\exists N$: $a_n \le b_n$ for $n > N$. Then $a \le b$.

Proof.

Let $a > b$. Take $\varepsilon = \frac{a-b}{2} > 0$. All but finitely many a_n fall into ε-neighborhood of a, so $a_n > a - \varepsilon = \frac{a+b}{2}$. Similarly, all but finitely many b_n fall into ε-neighborhood of b, so $b_n < b + \varepsilon = \frac{a+b}{2}$. So for all but finitely many n, $a_n > b_n$ - contradiction.

Theorem.

Let $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$. Then

1. $\lim_{n\to\infty}(a_n \pm b_n) = a \pm b$
2. $\lim_{n\to\infty}(a_n b_n) = ab$
3. $\lim_{n\to\infty}(a_n/b_n) = \frac{a}{b}$ if $b \ne 0$.

Proof - see the book.

Examples.

1. $\lim_{n\to\infty} \frac{n^4 + 3n^2 - 2n + 5}{n^4 - 5n} = 1$
2. $\lim_{n\to\infty} \sqrt{n}\left(\sqrt{n+2} - \sqrt{n+1}\right) = {}^1/_2$
3. $\lim_{n\to\infty} \sqrt[3]{n^2}\left(\sqrt[3]{n+1} - \sqrt[3]{n-1}\right) = {}^2/_3$

Supremum and infimum

Reminder: upper and lower bounds. *Sic:* the inequalities are not strict. An upper/lower bound can belong to a set!
Observation: m is an upper bound for a set S iff $-m$ is a lower bound for a set $-S. = \{x : -x \in S\}$. S is bounded above iff $-S$ is bounded below.
Observation: upper bounds form a ray: if $m \geq n$, and n is an upper bound for S, then m is also an upper bound for S.
Definition.
Let S be a bounded above set. L is called *the least upper bound* or *supremum* of S if L is an upper bound for S, and for any other upper bound m of S we have $m \geq L$.
Notation: $L = \sup S$.
Equivalently: $L = \sup S$ iff L is an upper bound for S, and **any** $m < L$ is **not** an upper bound for S.
Equivalently: $L = \sup S$ iff L is an upper bound for S, and $\forall \varepsilon > 0 \, \exists s \in S : s > L - \varepsilon$.

Definition.
Let S be a bounded above set. L is called *the greatest lower bound* or *infimum* of S if L is a lower bound for S, and for any other lower bound m of S we have $m \leq L$.
Notation: $L = \inf S$.
Observation: $\sup S = -\inf(-S)$.
Theorem (Least upper bound principle).
Every nonempty subset S of $\mathbb{R}$ that is bounded above has a supremum. Similarly, every nonempty subset S of $\mathbb{R}$ that is bounded below has an infimum.
Proof.
As in the book, we give a proof for the infimum. The existence of the supremum follows from the observation above.
First let us observe that a limit of a sequence of lower bounds for S is again a lower bound -- we just need to check it for *every* $s \in S$.
Next, we can always find an *integer* lower bound for S, let us call it m. Then 0 is a lower bound for the set $S - m$. (We work with the set $S - m$ since it consists of nonnegative numbers - 0 is its lower bound!).
Let s be some element of S with decimal expansion $s = s_0.s_1s_2\dots$. Notice that $s_0 + 2$ is *not* a lower bound for S. (Why do we have to add 2?)
There are only finitely many integers between 0 and s_0+1. Pick the largest of these that is still a lower bound for S, and call it a_0. Notice that $a_0 + 1$ is again *not* a lower bound for S.
Next pick the greatest integer a_1 such that $y_1 = a_0 + 10^{-1}a_1$ is a lower bound for S. Since $a_1 = 0$ works and $a_1 = 10$ does not, a_1 belongs to $\{0,1,\dots,9\}$. Notice that $y_1 + 10^{-1}$ is not a lower bound for S.
Same way we construct the number L=$a_0.a_1a_2 \dots$, such that for each n, $y_n = a_0.a_1a_2 \dots a_n$ a lower bound for S, but $y_n + 10^{-n}$ is not.
Since $L = \lim y_n$, L is a lower bound for S. On the other hand, if $m > L$, then $m > L + 10^{-n} \geq y_n + 10^{-n}$ for some n, so m is not lower bound for S.
Thus $L = \inf S$.
□

Monotone sequences

Definition.
A sequence of real numbers $(a_n)$is called (strictly) increasing if $a_{n+1} \geq a_n$ $(a_{n+1} > a_n)$.
A sequence of real numbers $(a_n)$is called (strictly) decreasing if $a_{n+1} \leq a_n$ $(a_{n+1} < a_n)$.
A sequence which is either (strictly) increasing or (strictly) decreasing is called *(strictly) monotone*.

Theorem. (Monotone convergence theorem).
Every bounded above increasing sequence converges to its supremum.
Every bounded bellow decreasing sequence converges to its infimum.
Proof.
Enough to prove for increasing sequences. Let $L = \sup a_n$. Then
$\forall \varepsilon > 0 \exists a_N : L \geq a_N > L - \varepsilon$.
Since a_n is increasing,
$\forall n > N : L \geq a_n \geq a_N > L - \varepsilon$.■
Remark.
The unbounded increasing (decreasing) sequences diverge to $+\infty$ $(-\infty)$. The proof is the same as for Monotone Convergence Theorem.
Examples.
1. $a_1 = 1, a_{n+1} = \sin a_n$. Bounded decreasing sequence, hence converges to 0.
2. $a_1 = .001, a_2 = 0.1$, $a_{n+2} = a_{n+1} + \frac{a_n^2}{100}$. This sequence is unbounded!
3. Important!: (a_n) is a sequence of nonnegative numbers, $s_n := \sum_{k=1}^{n} a_k$. Then s_n either have a limit, or diverges to $+\infty$.

Lemma (Nested Intervals Lemma).
Suppose that $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$ are nonempty closed intervals such that $I_{n+1} \subset I_n$ for each n 1. Then the intersection $\cap_{n\geq 0} I_n$ is nonempty.
Proof.
(a_n) is an increasing sequence, and (b_n) is decreasing. Let a be the limit of (a_n) , and b be the limit of (b_n). Then $a \leq b$, and $\forall n$, $\{x: a \leq x \leq b\} \subset I_n$. ■
Remark.
It is important to consider **closed** intervals!
Corollary.
If $I_{n+1} \subset I_n$, and $|I_n| = b_n - a_n \to 0$, then $\cap_{n\geq 0} I_n$ consists of exactly one point.
Proof.
Assume that $c, d \in \cap_{n\geq 0} I_n$. Then $\forall n, |d - c| \leq b_n - a_n$, so $d = c$. ■

Cauchy sequences

Definition.
A sequence of real numbers (a_n) is called a **Cauchy sequence** if
$\forall \varepsilon > 0 \exists N$: if $n, m > N$, then $|a_n - a_m| < \varepsilon$.
Lemma.
Every converging sequence is a Cauchy sequence.
Proof.
Fix $\varepsilon > 0$ and let $a_n \to L$. Then one can find N, such that for $n > N$ $|a_n - L| < \varepsilon/2$.
Then when $n, m > N$, $|a_n - a_m| \le |a_n - L| + |L - a_m| < \varepsilon$. ■
Turns out that for $\mathbb{R}$ the opposite is also true. Not so for $\mathbb{Q}$!

Completeness Theorem.
Every Cauchy sequence of real numbers have a limit.
Proof.
Let (a_n) be a Cauchy sequence.
First, take N such that if $n, m > N$, then $|a_n - a_m| < 1$. Since
$\{a_n, n \in \mathbb{N}\} \subset (a_{N+1} - 1, a_{N+1} + 1) \cup \{a_1, a_2, \dots, a_N\}$
(a_n) is a bounded sequence.

Let $u_k \coloneqq \sup\{a_n, n \ge k\}, l_k \coloneqq \inf\{a_n, n \ge k\}$.
Note that $u_k \ge l_k$, so both sequences are bounded.
Use the Monotone sequence Theorem to define $L_+ \coloneqq \lim u_k$, $L_- \coloneqq \lim\ l_k$
Remark. The construction works for any **bounded** sequence.
Notation.
$L_+ \coloneqq \limsup a_n$, $L_- \coloneqq \liminf a_n$

Moreover, $\forall \varepsilon > 0 \exists N: m > N \Rightarrow |a_{N+1} - a_m| < \varepsilon/2$.
So for $k > N$: $|a_{N+1} - u_k| \le \varepsilon/2$ and $|a_{N+1} - l_k| \le \frac{\varepsilon}{2}$. So $u_k - l_k \le \varepsilon$.
Thus $u_k - l_k \to 0$. So, $\forall \varepsilon > 0$, $L_+ - L_- \le \varepsilon$. Thus $L_+ = L_- =: L$.

Note that $u_k \ge a_k \ge l_k$, and $u_k \to L$, $l_k \to L$.
By Squeezed Sequences Theorem, $a_k \to L$. ■
Example (important).
Majorated convergence of series: $\sin n / 2^n$.

Theorem.
If $|a_n| \le M_n$, and $\sum M_n$ converges, so does $\sum a_n$.
Proof.
Follows from 2.8.C■

Series

Lemma.
If $|q| < 1$, the series $\sum_{n=0}^{\infty} q^n$ converges, and
$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$$
Proof.
$$s_k = \frac{1-q^{k+1}}{1-q} \to \frac{1}{1-q}. \blacksquare$$
Lemma.
If $\sum_{n=0}^{\infty} a_n$ converges, then $a_n \to 0$.
Proof.
$$a_n = s_n - s_{n-1} \to \sum_{n=0}^{\infty} a_n - \sum_{n=0}^{\infty} a_n = 0. \blacksquare$$

Observation (from the definition of the limit).
$\sum_{k=0}^{\infty} a_k$ exists if and only if $\forall \varepsilon > 0 \exists N\colon\ n > N \Rightarrow |\sum_{k=n}^{\infty} a_k| < \varepsilon$.
Restatement of Cauchy Theorem for series -- Cauchy Criterion.
$\sum_{k=0}^{\infty} a_k$ exists if and only if $\forall \varepsilon > 0 \exists N\colon n, m > N \Rightarrow |\sum_{k=n}^{m} a_k| < \varepsilon$.
Theorem.
$\sum_{k=1}^{\infty} \frac{1}{k^\alpha} < \infty$ iff $\alpha > 1$.
Special case: when $\alpha = 1$, the *harmonic series* $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.
Proof.
Let $\alpha \le 1$. By comparison test, enough to prove that harmonic series diverges.
Note that for harmonic series
$$s_{2^{n+1}} - s_{2^n} = \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k} \ge 2^n \frac{1}{2^{n+1}} = \frac{1}{2}.$$
This implies convergence by Cauchy criterion.
On the other hand, if $\alpha > 1$,
$$s_{2^{n+1}} - s_{2^n} = \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k^\alpha} \le 2^n \frac{1}{2^{n\alpha}} = \left(\frac{1}{2}\right)^{(\alpha-1)n}.$$
Since $\left(\frac{1}{2}\right)^{(\alpha-1)} < 1$, we get that $s_{2^n} \le \sum_{k=0}^{n-1} \left(\frac{1}{2}\right)^{(\alpha-1)k} < \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{(\alpha-1)k} < \infty$.
So $\{s_{2^n}\}$is bounded. It means that $\{s_n\}$ is also bounded. $\blacksquare$
Root test for convergence.
Let $a_n \ge 0$. Define $L \coloneqq \limsup \sqrt[n]{a_n}$.
If $L > 1$, the series $\sum_{n=1}^{\infty} a_n$ diverges.
If $L < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges.
Remark.
If $a_n = {}^1/_{n^\alpha}$, $L = 1$, yet for $\alpha > 1$ the series $\sum_{n=1}^{\infty} a_n$ converges, and for $\alpha \le 1$ the series $\sum_{n=1}^{\infty} a_n$ diverges.
Remark - a property of $\limsup$.
$\forall \varepsilon > 0$: $\{n\colon a_n > (L+\varepsilon)^n\}$ is finite and $\{n\colon a_n > (L-\varepsilon)^n\}$ is infinite.
Proof of the root test.
Let $L < 1$. Pick $\varepsilon > 0$, so that $L + \varepsilon < 1$. Then for all but finitely many n, $a_n < (L+\varepsilon)^n$, so a_n converges by comparison test.
Let now $L > 1$. Pick $\varepsilon > 0$, so that $L - \varepsilon > 1$. Then $\forall N \exists n > N\colon a_n > (L-\varepsilon)^n > 1$,

so $a_n \nrightarrow 0$. ■

Ratio test for convergence.

Let $a_n > 0$. Define $U := \limsup \frac{a_{n+1}}{a_n}$, L:= $\liminf \frac{a_{n+1}}{a_n}$.

If $L > 1$, the series $\sum_{n=1}^{\infty} a_n$ diverges.

If $U < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges.

Proof.

Homework■

Theorem (Leibniz Alternating Series).

Let $a_n \geq 0$ be a monotone decreasing sequence converging to zero. Then $\sum_{n=0}^{\infty} (-1)^n a_n$ converges.

Proof.

Note that $s_{2n} - s_{2n+2} = a_{2n+1} - a_{2n+2} > 0$ and $s_{2n+1} - s_{2n-1} = a_{2n} - a_{2n+1} > 0$. Thus the sequence s_{2n} is decreasing, and the sequence s_{2n-1} is increasing. Moreover, $s_{2n} - s_{2n-1} = a_{2n} > 0$. Since $a_{2n} \to 0, \lim(s_{2n} - s_{2n-1}) = 0$.

Both sequences s_{2n} and s_{2n-1} are bounded (by each other), so they both have limits, which are the same by the previous observation.

■

Limit points and subsequences

Definition.
Let (x_n) be a real sequence. $x \in \mathbb{R}$ is called a *limit point* of (x_n) if
$\forall \varepsilon > 0 \; \{n \in \mathbb{N}: \; |x_n - x| < \varepsilon\}$ is infinite.
Remark
Compare with the limit, where the set $\{n \in \mathbb{N}: \; |x_n - x| < \varepsilon\}$ should contain all but finitely many points.
Examples.
1. $(-1)^n$ has two limit points: {-1,1}
2. Any converging sequence has only one limit point, its limit.
3. $\mathbb{N}$ (viewed as a sequence) has no limit points
4. The sequence $x_n = \begin{cases} 1, n \text{ odd} \\ n, n \text{ even} \end{cases}$ has only one limit point: 1.

Definition.
Let n_k be an increasing sequence of natural numbers. (x_{n_k}) is called *a* subsequence of sequence (x_n).
Easy to see by induction: $n_k \geq k$
Theorem.
x is a limit point of a sequence (x_n) iff $x = \lim_{k\to\infty} x_{n_k}$ for some subsequence of (x_n).
Proof.
If $x = \lim_{k\to\infty} x_{n_k}$ then $\{n \in \mathbb{N}: \; |x_n - x| < \varepsilon\} \supset \{n_k \in \mathbb{N}: \; |x_{n_k} - x| < \varepsilon\}$, which is infinite by the definition of the limit.
On the other hand, if x is a limit point of x_n , we can construct a subsequence x_{n_k} recursively, by selecting $n_{k+1} > n_k$, with $|x_{n_k} - x| < \frac{1}{k}$. ■
Theorem. (Bolzano-Weierstrass)
Every bounded sequence has a converging subsequence.
Equivalently: every bounded sequence has a limit point.
Proof.
Let us choose intervals $[a_k, b_k]$ recursively, so that
1. $\{x_n\} \subset [a_1, b_1]$ (can be done because (x_n) is bounded)
2. $[a_k, b_k]$ is either left or right half of $[a_{k-1}, b_{k-1}]$, so that $\{n: \; x_n \in [a_k, b_k]\}$ is infinite . This is done by induction.

Then $|b_k - a_k|$ =$|b_{k-1} - a_{k-1}|$ /2, so $|b_k - a_k| \to 0$.
Moreover, the family $([a_k, b_k])$ is nested. Thus
$\bigcap_k [a_k, b_k] = \{x\}$.
Then $\forall \varepsilon > 0: \exists k: [a_k, b_k] \subset (x - \varepsilon, x + \varepsilon)$.
So $\{n: \; x_n \in [a_k, b_k]\} \subset \{n \in \mathbb{N}: \; |x_n - x| < \varepsilon\}$.
So x is a limit point of x_n. ■